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NUMERICAL ANALYSIS OF THE ASYMPTOTIC REPRESENTATION OF SOLITARY WAVES

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Almost all the published numerical and analytical studies of solitary waves on the surface of a liquid consider only one solution, although a non-uniqueness theorem has been proven recently [1]. A method that allows all the solutions to be constructed has been proposed [2]. The essence of this method is to seek a series solution, in which the terms are found from recurrence formulas but the first term remains undetermined. The equation found for the first term can have several solutions.

Herewerealizethis method numerically, examine branching of the solution, and compareitwiththeworkofprior authors.

The initial studies [3-6] of a solitary wave on the surface of a liquid gave rise to the soliton topic. Most of the solutions are approximations for low-amplitude solitary waves. There are many fewer exact results, including the proof of the existence of solitary waves [7, 8], the non-uniqueness of solitary waves for a fixed Froude number [1], and the existence of a sharp peak with a 120° vertex angle for a wave of limiting amplitude [9]. The shape of the wave profile, its mass, energy, and momentum, etc. are mainly obtained numerically. Two groups of methods are used. The first reduces the problem to an integro-differential equation and uses a finite-difference solution [10, 11]. The second numerically sums a solution represented in the form of a series [12-14].

We now examine the plane irrotational steady-state flow of a heavy liquid over a horizontal bottom. The X-axis of the Cartesian coordinate system is along the bottom, while the Y-axis is vertical upwards. The origin of the coordinates is located on the bottom, such that the Y-axis passes through the highest point of the free surface (Fig. 1). Here h_0 is the depth of the unperturbed liquid at infinity, u_0 is the velocity of the incoming flow at infinity, g is the acceleration due to gravity, φ is the potential, and ψ is the streamline function.

The problem of finding the solitary wave, i.e. constructing a flow with a free boundary $Y = Y_1(X)$, which is satisfied by the condition

$$\lim_{|X| \to \infty} Y_1(X) = h_0,$$

depends on a single parameter, for which we can use the Froude number

$$\mathrm{Fr} = \frac{u_0}{\sqrt{gh_0}} > 1$$

or the Stokes parameter θ, which can be determined from the equation

$$\mathrm{Fr}^2 = \frac{\mathrm{tg}\theta}{\theta},\ 0 \leq \theta < \frac{\pi}{2},$$

which arises after the initial problem is linearized for the uniform flow problem: $Y_1(X) = h_0$ and $\varphi = u_0X$. This linearization is valid when the free surface differs insignificantly from the unperturbed level and evidently remains locally valid as $|X| \to \infty$. In the primary term,

$$Y_1(X) \sim h_0\exp(-\theta|X|/h_0).$$

Low-amplitude solitary waves correspond to the limit $\theta \to 0$. The Stokes parameter $\theta^* \approx \pi/3$ corresponds to a solitary wave of limiting amplitude. Numerical results [11] show that a Stokes parameter θ^{**} ($\theta^* < \theta^{**} < \pi/2$) exists such that there is a single wave for $0 \leq \theta < \theta^*$, two waves for $\theta^* \leq \theta < \theta^{**}$, but no solitary waves for $\theta > \theta^{**}$. It is possible that the non-

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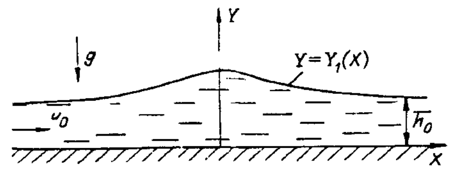

Fig. 1

uniqueness shown in [1] is explained by the fact that θ is not a unique parameter. If the Froude number is determined by the velocity at the apex of the wave, the non-uniqueness vanishes [14].

A solitary wave is described by higher approximations of shallow-water theory, which has been derived systematically by Friedrichs [15]. The results can be represented by a power series in θ^2 (or the amplitude). Based on the first [16], second [17], and higher approximations obtained later, it can be assumed that

$$\frac{Y_1(X)}{h_0} = 1 + \sum_{j=1}^{\infty}\theta^{2j}\sum_{n=1}^{j}\frac{a_{jn}}{\left(\operatorname{ch}\frac{\theta X}{2h_0}\right)^{2n}}. \tag{1}$$

Evidently no one has proven why the coefficients of this series should be polynomials in $[\cosh\{\theta X/2h_0\}]^{-2}$, although this form is often used, and efforts are devoted to maximizing the number of terms in this series. This is difficult to do without a computer, because the recurrence formulas are very complex. With a computer, solutions to Eq. (1) have been found to θ^{18} [13], θ^{28} [14], and θ^{34} [12]. If the sums in (1) are transposed, then

$$\frac{Y_1(X)}{h_0} = 1 + \sum_{n=1}^{\infty}\frac{a_n}{\left(\operatorname{ch}\frac{\theta X}{2h_0}\right)^{2n}}, \tag{2}$$

where

$$a_n = O(\theta^{2n}). \tag{3}$$

If this series is substituted into the boundary conditions, we obtain recurrence formulas for sequentially finding the coefficients a_n. This was done by Pennel and Su [12]; however no solution in the form of (2) was found, because the first coefficient a_1 remains undetermined.

The problem is significantly simplified if it is formulated, not in the physical X−Y plane, in which the free boundary is unknown, but in the plane of the complex potential $\Phi = \varphi + i\psi$. We make the problem dimensionless, by following Ovsyannikov [2]:

$$Z = X + iY = \frac{h_0}{\theta}(z + W(z)),\ z = x + iy = \frac{\theta}{h_0 u_0}\Phi. \tag{4}$$

The liquid occupies a strip of width θ in the plane of the dimensionless complex potential z. The solitary wave is found by solving the following problem:

Problem 1. Find the function $W(z) = A(x,y) + iB(x,y)$, which is analytic in the strip

$$0 < y < \theta,\ -\infty < x < \infty,$$

which satisfies the condition

$$\lim_{|x|\to\infty} B(x,y) = 0, \quad A(x,y) = O(1)\ npu\ |x| \to \infty$$

and the boundary conditions [2]:

$$B_y - \nu B = f \quad (y = \theta)$$
$$\left(f = \frac{2\nu^2 B^2}{1 - 2\nu B} - \frac{1}{2}(B_x^2 + B_y^2),\ \nu = \mathrm{ctg}\theta\right); \tag{5}$$

for constant pressure and

$$B = 0 \quad (y = 0). \tag{6}$$

for no flow.

A solution to the nonlinear boundary problem exists, but it is difficult to find (and evidently can only be found numerically). Therefore it was suggested that a simpler problem [2] be solved.

Problem 2. Find the function represented by the asymptotic series

$$W(z) = \sum_{n=1}^{\infty} b_n \int_0^z \mu^n dz,\ \mu = \frac{1}{2\left(\mathrm{ch}\frac{z}{2}\right)^2},\ \mathrm{Im} b_n = 0, \tag{7}$$

which satisfies the boundary condition (5).

By substituting the series (7) into the boundary condition (5), one can obtain recurrence formulas for sequentially finding the coefficients b_n. As in (2), the first coefficient b_1 remains undetermined. Here we find b_1 by solving the determining equation in [2] numerically. Evidence has been found to support the fact that one of the solutions constructed in this manner coincides with (1).

The higher approximations for shallow water are constructed in the physical plane by deforming the horizontal variable X while keeping the vertical variable Y constant [15]. This makes the functions which describe the flow non-analytic. Ovsyannikov [2] found a way to formulate the problem in the plane of the complex potential such that the functions remain analytic. The X-axis must be deformed by somewhat artificially normalizing (4) so that the strip in the z plane which corresponds to the liquid is deformed to a line as $\theta \to 0$. The higher approximations of the shallow-water theory are constructed by solving the following problem.

Problem 3. Find the function, represented by the formal power series

$$W(z) = \sum_{j=1}^{\infty} W^{(j)}(z)\theta^{2j}, \tag{8}$$

which satisfies the boundary conditions (5) and (6).

By substituting (8) into the boundary condition, taking the limit $\theta \to 0$, and collecting like terms in powers of θ, we obtain

$$W^{(1)}_{zzz} - W^{(1)}_z + \frac{9}{2}(W^{(1)}_z)^2 = 0.$$

for θ^4. The solution to this equation that vanishes at infinity is

$$W^{(1)}_z = \frac{2}{3}\mu,\ W^{(1)} = \frac{2}{3}\mathrm{th}\frac{z}{2},\ \mu = \frac{1}{2\mathrm{ch}^2\frac{z}{2}}.$$

Analysis of the subsequent coefficients is much simpler here than for (1), and it turns out that all the $W^{(j)}_z$ are polynomials in μ. Thus the choice of the decomposition of (7) can be understood. It is in the same correspondence with (8) as the decomposition of (2) is with (1). Now we can forget about the derivation of the series (7) by seeking a solution in the form of (7) in the hope that this series also describes other solutions.

We now examine the imaginary part of (7). By following Ovsyannikov [2] we obtain

$$B(x,y) = \sum_{n=1}^{\infty} b_n B_n(x,y), \tag{9}$$

where

$$B_n(x,y) = \frac{1}{2^{n-1}} \mathrm{Im} \int_0^{\chi(z)} (1 - u^2)^{n-1} du;$$

$$\chi(z) = \frac{e^z - 1}{e^z + 1} = Q + iP; \; P = \frac{\sin y}{\mathrm{ch} x + \cos y}.$$

The functions B_n are polynomials in P, whose coefficients in tern are polynomials in $\omega = \cot y$. We have

$$B_1 = P,$$

$$B_2 = \omega P^2 + \frac{2}{3} P^3,$$

$$B_3 = \left(\omega^2 - \frac{1}{3}\right) P^3 + 2\omega P^4 + \frac{4}{5} P^5,$$

......

$$B_n = \sum_{k=n}^{2n-1} \frac{1}{k} P^k \sum_{l=\left[\frac{k}{2}\right]}^{n-1} (-1)^{n-l-1} \binom{k}{2l+1-k} \binom{k-1-l}{n-1-l} 2^{k-n} \omega^{2l+1-k}.$$

We substitute (9) into (5) and, by using the formulas

$$P_y = \omega P + P^2, \; P_x^2 + P_y^2 = (\omega^2 + 1) P^2,$$

we transform the boundary condition into a power series in P. We equate terms for the same power of P (the first term is an identity), and obtain a recurrent chain of formulas for sequentially finding the b_n. The coefficient b_1 is undetermined, while the others are functions of it:

$$b_2 = b_1 + \left(-\frac{3}{2}\nu^2 + \frac{1}{2}\right) b_1^2; \tag{10}$$

$$b_3 = \frac{5}{4} b_1 + (-3\nu^2 + 1) b_1^2 + \left(\frac{27}{16}\nu^4 - \frac{21}{8}\nu^2 + \frac{3}{16}\right) b_1^3; \tag{11}$$

$$b_4 = \frac{7}{4} b_1 + \left(-\frac{21}{4}\nu^2 + \frac{7}{4}\right) b_1^2 + \left(\frac{81}{16}\nu^4 - \frac{63}{8}\nu^2 + \frac{9}{16}\right) b_1^3 + \frac{1}{5\nu^2 - 1}\left(-\frac{135}{16}\nu^8 + \frac{261}{8}\nu^6 - \frac{41}{2}\nu^4 + \frac{19}{8}\nu^2 - \frac{1}{16}\right) b_1^4; \tag{12}$$

....

$$b_n(b_1) = \sum_{k=1}^{n} d_n^k b_1^k \;. \tag{13}$$

From (12) it can be seen that b_4 becomes infinite when $\nu^2 = 1/5$ or $\mathrm{tg}\theta = \sqrt{5}$. Moreover, this occurs for an infinite number of values of θ, which satisfy the equation

$$\mathrm{tg} m\theta = m \,\mathrm{tg}\theta, \; m \geqslant 4. \tag{14}$$

If we find the value of m for which this equation is satisfied for a given θ, we can then set

$$b_1 = b_2 = \ldots = b_{m-1} = 0$$

and seek a solution in the form

$$B = \sum_{n=m}^{\infty} b_n B_n(x,y).$$

Here b_m appears as an undetermined coefficient.

Following Ovsyannikov [2], we examine the expression

$$\lim_{x\to\infty} \int_0^\theta (BH_x - HB_x)dx \quad (H = \text{ch}x \cdot \sin y).$$

This is $b_1(\theta - \sin\theta\cdot\cos\theta)$ because of the asymptote $B = b_1P + O(P^2)$ on one hand; on the other hand it is

$$\int_0^a (Hf)_{y=\theta}dx = \int_0^\theta (BH_x - HB_x)_{x=a}dy,$$

due to Green's formula and the boundary conditions. By examining the limit as $a \to \infty$, we have

$$G(b_1) = \int_0^\infty f(x,\theta)\text{ch}x dx - b_1\left(\frac{\theta}{\sin\theta} - \cos\theta\right) = 0, \tag{15}$$

where f is the function in the boundary condition (5). This equation, obtained by Ovsyannikov [2], is not an identity and should be satisfied by any solution of problem 1. Therefore it can be used to find b_1.

A numerical solution of the equation $G(b_1) = 0$ must consider the fact that f is not known exactly. It can be represented as a series and this series must be truncated. Here the function f is represented as a series in powers of P. By truncating this series at the N-th term, we obtain the approximate equation $G_N(b_1) = 0$. The value of b_1 can be found by observing how the root of this equation behaves as $N \to \infty$.

By substituting (9) into (15) we have the equation

$$G(b_1) = \sum_{n=2}^{\infty} b_n(b_1)\int_0^\infty L_n \text{ch}x dx. \tag{16}$$

Here the function

$$L_n = \frac{\partial}{\partial y}B_n - \omega B_n\Big|_{y=\theta}$$

is a polynomial in P and ν:

$$L_1 = P^2,$$

$$L_2 = -P^2 + \frac{10}{3}\nu P^3 + 2P^4,$$

$$\ldots\ldots$$

$$L_n = \sum_{j=n}^{2n} a_n^j P^j .$$

By direct integration, it is not difficult to show that

$$\int_0^\infty L_n \text{ch}x dx \equiv 0, \ n \geqslant 2.$$

Therefore Eq. (16), which can be written more explicitly as

$$G(b_1) = \sum_{n=2}^{\infty} b_n(b_1)\sum_{j=n}^{2n} a_n^j \int_0^\infty P^j \text{ch}x dx \equiv 0, \tag{17}$$

actually is an identity, which is valid for any b_1. Here the double series is evidently not absolutely convergent, because changing the order of summation, which must be done to obtain $G_N(b_1)$, transforms the identity into the equation

$$G(b_1) = \sum_{j=2}^{\infty} \int_0^\infty P^j \text{ch}x Q_j dx \quad \left(Q_j = \sum_{k=1}^{j} b_1^k \sum_{n=\max\left(2,k,\left[\frac{j+1}{2}\right]\right)}^{j} a_n^j d_n^k\right).$$

Here d_n^* is determined from (13). The first few initial terms are

$$Q_2 = b_1^2\left(\frac{3}{2}\nu^2 - \frac{1}{2}\right) - b_1,$$

$$Q_3 = b_1^3\left(-\frac{9}{2}\nu^5 + 7\nu^3 - \frac{1}{2}\nu\right) + b_1^2(3\nu^3 - \nu),$$

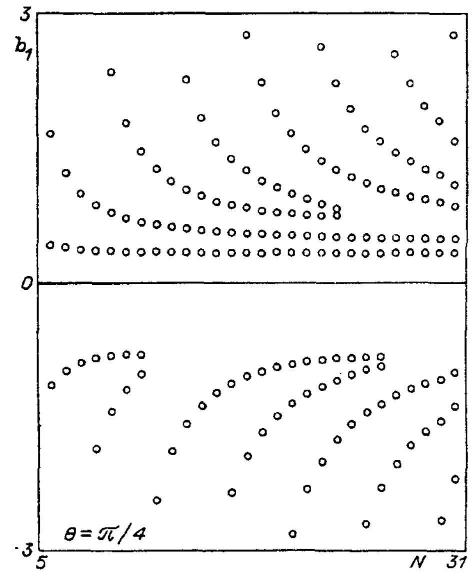

Fig. 2

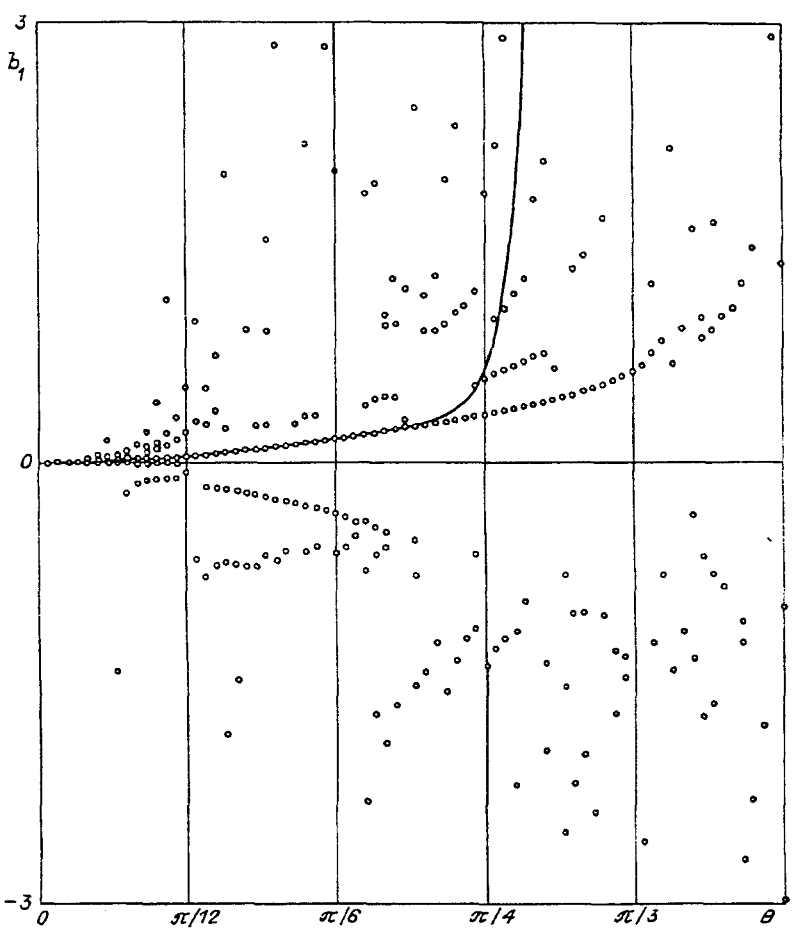

Fig. 3

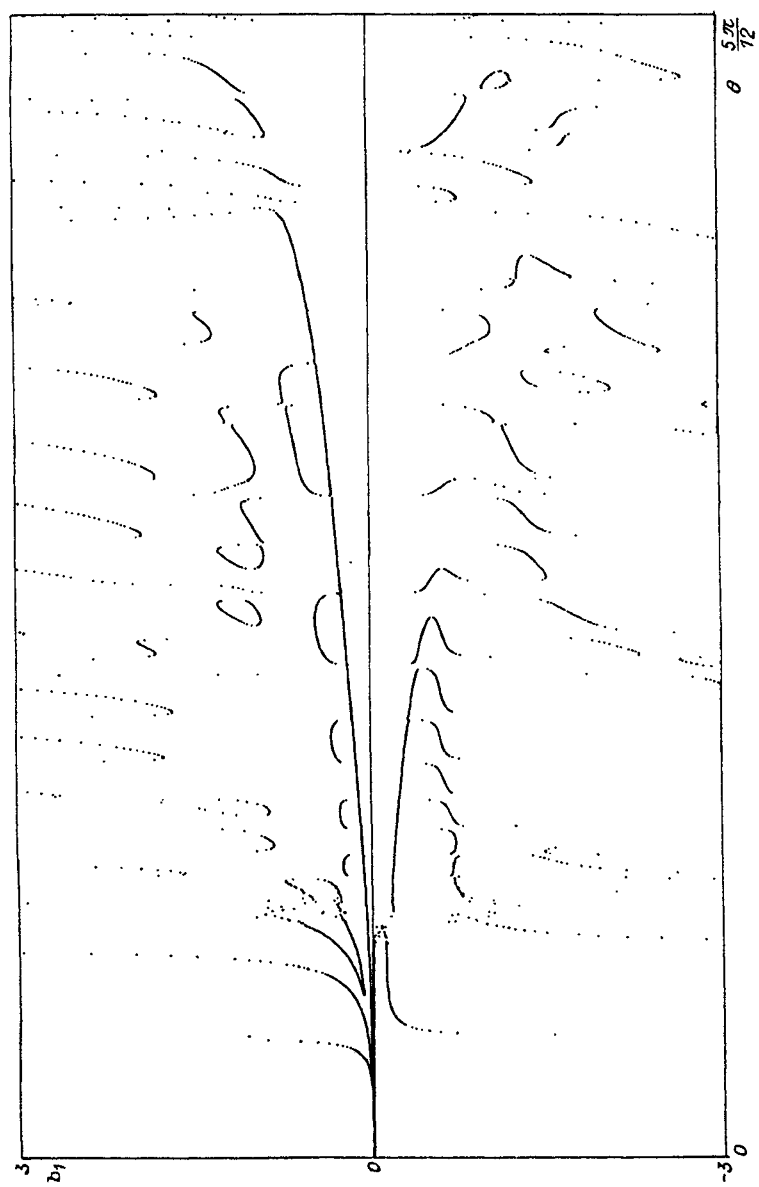

Fig. 4

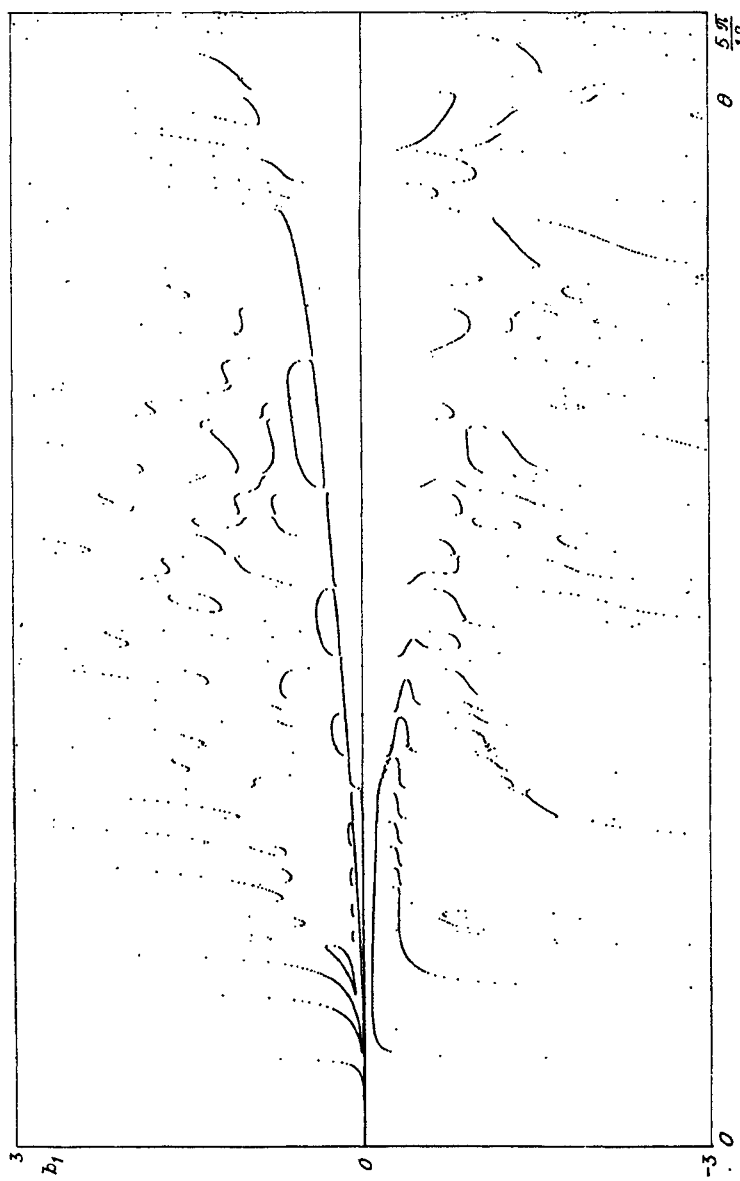

Fig. 5

$$Q_4 = b_1^4\left(\frac{135}{16}\nu^8 - \frac{261}{8}\nu^6 + \frac{41}{2}\nu^4 - \frac{19}{8}\nu^2 + \frac{1}{16}\right)$$
$$+ b_1^3\left(-\frac{27}{2}\nu^6 + 21\nu^4 - \frac{3}{2}\nu^2\right) + b_1^2\left(\frac{21}{4}\nu^4 - \nu^2 - \frac{1}{4}\right).$$

We also have

$$\int_0^\infty P^j \mathrm{ch}x dx = \sin\theta \cdot K_j,$$
$$K_2 = \theta(\nu^2 + 1) - \nu;$$
$$K_3 = \theta\left(-\frac{3}{2}\nu^3 - \frac{3}{2}\nu\right) + \frac{3}{2}\nu^2 + 1;$$
$$K_4 = \theta\left(\frac{5}{2}\nu^4 + 3\nu^2 + \frac{1}{2}\right) - \frac{5}{2}\nu^3 - \frac{13}{6}\nu.$$

Thus, the defining Eq. (15) is represented as a series of polynomials in b_1:

$$G(b_1) = \sum_{j=2}^{\infty}\sum_{k=1}^{j} g_j^k b_1^k .$$

The polynomial coefficients g_j^k are functions of θ. By truncating this series we obtain the approximate defining equation which is used here:

$$G_N(b_1) = \sum_{j=2}^{N}\sum_{k=1}^{j} g_j^k b_1^k = 0. \tag{18}$$

This equation has N roots. As N increases, more and more of them are unrelated to the problem. First of all, complex roots must be discarded (all negative roots can also be discarded, because they correspond to solitary depression waves which have been shown not to exist). Of the remaining real roots $b_1^{(N)}$, it is necessary to keep only those which satisfy the limit

$$\lim_{N\to\infty} b_1^{(N)} = b_1^*.$$

The calculations were done in rational numbers (IBM PC, REDUCE system, with 20 terms in the series) and in floating-point arithmetic (high-speed computer with a mantissa of 24 decimal digits, and 70 terms). Because of its weak mathematical foundation, the applicability of the algorithm must be verified numerically. The first question is, does a sequence of roots $\{b_1^{(N)}\}$ that has a limit exist in general? The answer is shown in Fig. 2, which shows all real roots of (18) in the range of $-3 \le b_1 \le 3$ for $5 \le N \le 31$ and $\theta = \pi/4$. It can be seen that at least two sequences of roots exist which tend to a limit. The second question which now arises is, does Ovsyannikov's method [2] describe the solution to (1) found earlier? To answer this question, values of b_1 obtained by solving Eq. (18) numerically must be compared with values obtained under the assumption that b_1 is an analytic function of θ^2 (or of $t = \tan^2\theta = \nu^{-2}$):

$$b_1 = \sum_{k=1}^{\infty} \beta_k t^k. \tag{19}$$

In analogy to (3), we require $b_n = O(t^n)$. By substituting (19) into Eq. (10) for b_2 and by setting terms linear in t to zero, we have $\beta_1 = 2/3$; then by substituting into Eq. (11) for b_3 and equating terms containing t^2 to zero, we obtain β_2 (terms linear in t vanish automatically). By continuing in this fashion, we find

$$b_1 = \frac{2}{3}t - \frac{2}{3}t^2 + \frac{262}{405}t^3 - \frac{6406}{10125}t^4 + \frac{1661986}{2679075}t^5 - \frac{612601582}{1004653125}t^6 + \dots \tag{20}$$

If the series (20) is substituted into (9), then the solution corresponds to (1). We note in passing that this method [2] of constructing the series is much simpler and cheaper than other algorithms that have been used [12-14]. In a few hours, an IBM PC could calculate 20 terms of the series (20), which exceeds the result obtained in [12].

If (20) is summed for $\theta = \pi/4$ (Pade summation [18] was used), then $b \simeq 0.33$. As can be seen from Fig. 2, this number corresponds exactly to the smaller value of b_1^*. The circles in Fig. 3 show all the real roots of $G_N(b_1)$ which lie in the interval $(-3\text{-}3)$ for $N = 15$ and $0 \leq \theta \leq 5\pi/12$. The solid lines show the sum of the series (20). There exists a family of roots which lies exactly on this line that starts to diverge from it at $\theta \simeq \pi/4$ because the radius of convergence of the series (20) is equal to unity.

We now represent the roots of the equation $G_N(b_1) = 0$ in the form of a power series
The coefficients of this series are expected to be approximately equal to the coefficients of the series (19):

$$\lim_{N \to \infty} \beta_k^{(N)} = \beta_k.$$

if N is large enough. Rational-number calculations show that there is an exact solution to the equation $\beta_k^{(n)} = \beta_k$, $k = 1, 2, \ldots, N$.

Thus, Ovsyannikov's method [2] describes the solution to (1). However, it also describes other solutions. The roots $G_N(b_1)$ for $N = 15$ and 20 are shown more precisely in Figs. 4 and 5 than they are in Fig. 3. Many of the roots can be discarded because they change with N; only those that do not change should appear in Figs. 4 and 5. The stability of the roots that correspond to the sum of the series (20) is immediately apparent. The horseshoe-shaped branches from this family are also stable. Evidently the solution branches here. It continues for those values of θ which satisfy Eq. (14).

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